Investment and production

- Our objective, eventually, is to have a general equilibrium model with both preferences and production (and maybe frictions or limited market structure) specified. But it makes sense to study preferences and production separately. Walk before you try to run. (See the graph of preferences, technology, and separating prices.)

Theory

- Here, I'll write down the simplest standard preferences, and let's see how they fit in to asset pricing. Analogously, the consumption problem is, maximize utility in contingent claim markets with an endowment stream e_t

$$\max E \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ s.t. } E_0 \sum_{t=0}^{\infty} \frac{\Lambda_t}{\Lambda_0}(c_t - e_t) = 0.$$

  The familiar answer is,

$$\beta \frac{u'(c_t)}{u'(c_0)} = \frac{\Lambda_t}{\Lambda_0}$$

  So, the question is, what happens if we do exactly the same sort of thing from the producer's side?

- Theory preview:

$$\begin{aligned} y_{t+1} &= \theta_{t+1} f(k_t) \\ \max_{\{k_t\}} E(m_{t+1} y_{t+1}) - k_t &= \max_{\{k_t\}} E(m_{t+1}\theta_{t+1} f(k_t)) - k_t \end{aligned}$$

$$\partial/\partial k : 1 = E(m_{t+1}\theta_{t+1} f'(k_t))$$

1. Interpretation 1: "price the investment return consistently with the other returns"

$$1 = E(m_{t+1} R^I_{t+1}(k_t))$$

2. Interepretation 2: marginal q

$$\text{mc of one unit of capital} = \frac{\partial V}{\partial k_t} = \frac{\partial}{\partial k_t} E(m_{t+1}\theta_{t+1} f(k_t))$$

3. With constant returns to scale,

$$f(k_t) = m_k k_t$$

   then we have two special additional results:

(a) Marginal q = average q, so invest until the marginal cost of investment = average q

$$\begin{aligned} \frac{V(k_t)}{k_t} &= \frac{E(m_{t+1}\theta_{t+1}m_k k_t)}{k_t} = E(m_{t+1}\theta_{t+1})m_k \\ 1 &= E(m_{t+1}\theta_{t+1})m_k \end{aligned}$$

Note, in this completely linear case, that means 0 or infinite investment; q always equals one. This is why we need some adjustment costs!

(b) Stock return = investment return, ex post.

$$R_{t+1}^{\text{stock}} = \frac{y_{t+1}}{V(k_t)} = \frac{\theta_{t+1}m_k k_t}{k_t}$$

- To a dynamic, serious model. I'll use the standard production function

$$\begin{aligned} y_t &= f(\theta_t, k_t, l_t) - c(i_t, k_t) \\ k_{t+1} &= (1-\delta)k_t + i_t \end{aligned}$$

1. $c(i,k)$ is an adjustment cost. Here, it is lost output – you don't get much work done while you're installing a new computer. My (1991) paper took it out of investment which is equivalent but a bit uglier.
2. For examples, I'll use a quadratic cost function

$$c(i,k) = \frac{\eta}{2}\left(\frac{i_t}{k_t}\right)i_t$$

There is a proportional cost, whose size depends on the intensity of installation. That has the right scaling properties. It is also homogenous of degree one, which is important below.

$$c_i = \eta\left(\frac{i_t}{k_t}\right); c_k = -\frac{\eta}{2}\left(\frac{i_t}{k_t}\right)^2$$

3. The central consequence of an adjustment cost is that *installed* capital is worth more than *uninstalled* capital. You have to pay the "installation" costs before new investment can become productive. Now stock prices – the value of the ownership claim to *installed* capital – can differ from the purchase cost of new capital goods. In a one-good model such as this, if there are no adjustment costs, the stock price is always one – the value of one unit of installed capital is just the value of one unit of the good. *Models without adjustment costs cannot hope to capture the fact that the relative price of stocks and other goods ever varies.*

- Firm objective:

$$\max E_0 \sum_{t=0}^{\infty} \frac{\Lambda_t}{\Lambda_0}(y_t - w_t l_t - i_t)$$

$$\begin{aligned} &\max E_0 \sum_{t=0}^{\infty} \frac{\Lambda_t}{\Lambda_0}\left[f(k_t, l_t) - c(i_t, k_t) - w_t l_t - i_t\right] \\ s.t.\ k_t &= (1-\delta)k_{t-1} + i_{t-1} \end{aligned}$$

Here the firm "owns" the capital stock. This is equivalent to usual formulations in which the firm "rents" capital for one period. None of this will make any difference – we'll end up with the same rules that a planner would use.

- Major result preview:

  1. The marginal cost of creating a new unit of installed capital – its' direct cost plus the marginal adjustment cost = marginal q = marginal change in the present value of profits coming from that investment

$$MC = 1 + c_i(i_t, k_t) = \frac{\partial V}{\partial k}$$

  2. The physical investment return R^I – invest \$1 more today, invest less tomorrow and collect extra output tomorrow, leave production, output, capital etc. unchanged for all other dates – should be priced consistently with other assets.

$$1 = E(mR^I).$$

  3. If the firm has constant returns to scale, then marginal q = average q, and the investment and stock returns should be equal ex post.

$$\frac{\partial V}{\partial k} = \frac{V}{k}; R^I_{t+1} = R^{\text{stock}}_{t+1}$$

- First order conditions. This is easy to do by iterating and substituting in the constraint

$$\max E_0 \sum_{t=0}^{\infty} \frac{\Lambda_t}{\Lambda_0} \left[f\left(\sum_{j=1}^{\infty} (1-\delta)^{j-1} i_{t-j}, l_t \right) - c\left(i_t, \sum_{j=1}^{\infty} (1-\delta)^{j-1} i_{t-j} \right) - w_t l_t - i_t \right]$$

Central Result 1

$$\partial/\partial i_0 : 1 + c_i(0) = E_0 \sum_{t=1} \frac{\Lambda_t}{\Lambda_0} (1-\delta)^{t-1} \left[f_k(t) - c_k(t) \right] \tag{13}$$

Here (t) means (k_t, i_t).

$$1 + \eta \left(\frac{i_0}{k_0} \right) = E_0 \sum_{t=1} \frac{\Lambda_t}{\Lambda_0} (1-\delta)^{t-1} \left[f_k(t) - c_k(t) \right]$$

- Intuition: The present value of future profits from investing \$1 should be \$1. To increase capital by \$1 without lowering output, you have to put in $1 + c_i(0)$ units of investment today. This turns in to 1 unit of capital tomorrow, $(1-\delta)$ units the day after, and so forth. $(1-\delta)^{t-1}$ units of capital means $(1-\delta)^{t-1} f_k(t)$ units more output via the production function, and also changes future adjustment costs by $(1-\delta)^{t-1} c_k(t)$. Thus $(1-\delta)^{t-1} [f_k(t) - c_k(t)]$ is the increase in profit at t from the investment at 0.

- *Q theory, marginal Q.*

1. *The quantity on the right hand side of the first order condition (13) is marginal q.* The firm's market value at the end of day 0 (after $y_0 - w_0 l_0 - i_0$ is paid out to shareholders) is

$$V_0(k_1, \cdot) = E_0 \sum_{t=1}^{\infty} \frac{\Lambda_t}{\Lambda_0} \left[f(k_t^*, l_t^*) - c(i_t^*, k_t^*) - w_t l_t^* - i_t^* \right]$$

where $*$ denotes optimal values (I won't keep using * below) and $\cdot$ denotes other state variables (not under the firm's control). (k_1 is known at time 0. This is simpler of course in continuous time.) Marginal q is defined as how much the value of the firm would rise if it had a bit more capital,

$$\frac{\partial V_0(k_1, \cdot)}{\partial k_1} = E_0 \sum_{t=1}^{\infty} \frac{\Lambda_t}{\Lambda_0} (1-\delta)^{j-1} \left[f_k(t) - c_k(t) \right].$$

Since $k_1 = (1-\delta)k_0 + i_0$, $\partial V / \partial k_1 = \partial V / \partial i_0$ which is the derivative we took.

2. The quantity on the left hand side is the extra capital you get from a dollar of investment, holding output y_0 fixed, which is typically a number less than one.

$$\frac{1}{1 + c_i(i_0, k_0)} = \left. \frac{\partial k_1}{\partial i_0} \right|_{y_0}$$

3. Thus, the first order condition says, *invest until the marginal cost of one extra unit of installed capital equals the marginal value – the marginal increase in the value of the firm – from an extra unit of installed capital.*

$$1 + \eta \frac{i_0}{k_0} = 1 + c_i(i_0, k_0) = 1 / \left. \frac{\partial k_1}{\partial i_0} \right|_{y_0} = \frac{\partial}{\partial k_1} V_0(k_1, \cdot)$$

If we could measure marginal q, we could solve this equation for i_0, telling us how much the firm should invest! (More when marginal Q is higher)

- *One-period first order condition and investment return.* We can also express the "one-period" first order condition by quasi-first differencing. (As we do to go from price = present value to return)

$$\begin{aligned}
1 + c_i(0) &= E_0 \left[\frac{\Lambda_1}{\Lambda_0} \left[f_k(1) - c_k(1) \right] \right] + E_0 (1-\delta) \frac{\Lambda_1}{\Lambda_0} \sum_{j=1} \frac{\Lambda_{1+j}}{\Lambda_1} (1-\delta)^{j-1} \left[f_k(1+j) - c_k(1+j) \right] \\
1 + c_i(0) &= E_0 \left[\frac{\Lambda_1}{\Lambda_0} \left[f_k(1) - c_k(1) \right] \right] + E_0 \left[\frac{\Lambda_1}{\Lambda_0} (1-\delta)(1 + c_i(1)) \right] \\
1 + c_i(0) &= E_0 \left[\frac{\Lambda_1}{\Lambda_0} (1-\delta) \left(1 + c_i(1)\right) + f_k(1) - c_k(1) \right] \\
1 &= E_0 \left[m_1 \frac{(1-\delta) \left(1 + c_i(1)\right) + f_k(1) - c_k(1)}{1 + c_i(0)} \right]
\end{aligned}$$

- *"Investment return"*

$$R^I_{t+1} = \frac{(1-\delta)\left[1+c_i(t+1)\right] + f_k(t+1) - c_k(t+1)}{1+c_i(t)}$$

$$R^I_{t+1}(i_t, k_t, i_{t+1}, k_{t+1}, l_{t+1}, \theta_{t+1}) = \frac{(1-\delta)\left[1+c_i(i_{t+1}, k_{t+1})\right] + f_k(\theta_{t+1}, k_{t+1}, l_{t+1}) - c_k(i_{t+1}, k_{t+1})}{1+c_i(i_t, k_t)}$$

1. Intuition: We can form a one period return by *lowering* investment at $t+1$ just enough so that there is no capital, output, etc. change at $t+2$ and beyond. In short *we create a one-period physical rate of return by investing a bit more at t and then a bit less at $t+1$, transferring resources from t to t+1, leaving all other dates unchanged.*
2. We measure the "investment return" from watching endogeous variables, via a production function, just as we measure the marginal rate of substitution $m_{t+1} = \beta\left(c_{t+1}/c_t\right)^{-\gamma}$ by watching endogeous variables, via a utility function.

- **Central Result 2:The producer first order conditions are**

$$1 = E_0\left[m_1 R^I_1\right]$$

The producer first order conditions say "the investment return should be correctly priced like any other asset return." If investment returns are a good deal, increase investment (or adjust other variables) until that's no longer true. Equivalently "remove any arbitrage opportunities between investment returns and returns of traded assets." This is the "first difference" of q theory.

- *Q theory, Average Q*

1. *If the production function, including the adjustment cost function is homogenous of degree one* (double all inputs doubles outputs) *then marginal q = average q = market/book ratio.*

$$\frac{\partial V(k_1, \cdot)}{\partial k_1} = \frac{V(k_1, \cdot)}{k_1}$$

2. Proof. Homogenous of degree one means

$$f(k,l) = f_k k + f_l l; \;\; c(i,k) = c_i i + c_k k$$

Then

$$\begin{aligned}
V_0(k_1, \cdot) &= E_0 \frac{\Lambda_1}{\Lambda_0}\left[f(k_1, l_1) - c(i_1, k_1) - w_1 l_1 - i_1\right] + \frac{\Lambda_t}{\Lambda_0} V_1(k_2, \cdot) \\
&= E_0 \frac{\Lambda_1}{\Lambda_0}\left[f_k(1)k_1 + f_l(1)l_1 - c_i(1)i_1 - c_k(1)k_1 - w_1 l_1 - i_1\right] + E_0 \frac{\Lambda_t}{\Lambda_0} V_1(k_2, \cdot) \\
\frac{V_0(k_1, \cdot)}{k_1} &= E_0 \frac{\Lambda_1}{\Lambda_0}\left[f_k(1) - c_k(1) - \left[1 + c_i(1)\right]\frac{i_1}{k_1}\right] + E_0 \frac{\Lambda_t}{\Lambda_0}\frac{k_2}{k_1}\frac{V_1(k_2, \cdot)}{k_2} \\
\frac{V_0(k_1, \cdot)}{k_1} &= E_0 \frac{\Lambda_1}{\Lambda_0}\left[f_k(1) - c_k(1) - \left[1 + c_i(1)\right]\frac{i_1}{k_1}\right] + E_0 \frac{\Lambda_t}{\Lambda_0}\frac{(1-\delta)k_1 + i_1}{k_1}\frac{V_1(k_2, \cdot)}{k_2} \\
\frac{V_0(k_1, \cdot)}{k_1} &= E_0\left\{\frac{\Lambda_1}{\Lambda_0}\left[f_k(1) - c_k(1) + \left[-\left[1 + c_i(1)\right] + \frac{V_1(k_2, \cdot)}{k_2}\right]\frac{i_1}{k_1} + (1-\delta)\frac{V_1(k_2, \cdot)}{k_2}\right]\right\}
\end{aligned}$$

If $\partial V/\partial k = V/k$, then$1 + c_i(1) = V_1(k_2,\cdot)/k_2$ and the second term cancels.

$$\frac{V_0(k_1,\cdot)}{k_1} = E_0\left\{\frac{\Lambda_1}{\Lambda_0}\left[f_k(1) - c_k(1) + (1-\delta)\frac{V_1(k_2,\cdot)}{k_2}\right]\right\}$$

Iterating, we have $V/k =$ marginal q.

3. **Major result 3** Now we can write

$$1 + \eta\frac{i_0}{k_0} = 1 + c_i(i_0, k_0) = \frac{V_0(k_1,\cdot)}{k_1} = \frac{\text{stock market value}}{\text{book value}}!$$

Firms should invest more when stock prices (q, m/b) are high, and invest less when prices are low.

4. *If there are no adjustment costs, q=B/M is always equal to one. To have any hope of matching the data, in which the vast majority of stock price variation does* **not** *correspond to variation in book value (quantity of capital) we need adjustment costs.* The fact that the vast majority of value fluctuation comes form changing prices of the same quantity of stuff should be central to all our thinking. For example, it makes a huge difference that the same quantity of houses now carries a lower price, rather than half the houses in the country just having blown up.

- **Major result 4** Stock return = investment return, ex-post. *When the technology is homogenous of degree one, the stock return should equal the investment return, ex-post.* This is really just a first differenced version of q theory. Q theory related investment to price; this statement relates investment growth to price growth (roughly speaking).

1. Algebra:

$$\begin{aligned}
R_2^{\text{stock}} &= \frac{V_1(k_2,\cdot) + [f(1) - c(1) - w_1 l_1 - i_1]}{V_0(k_1)} \\
&= \frac{(1 + c_i(1))\,k_2 + [f_k(1)k_1 + f_l(1)l_1 - c_i(1)i_1 - c_k(1)k_1 - w_1 l_1 - i_1]}{(1 + c_i(0))\,k_1} \\
&= \frac{(1 + c_i(1))\,((1-\delta)k_1 + i_1) + [f_k(1)k_1 - [1 + c_i(1)]\,i_1 - c_k(1)k_1]}{(1 + c_i(0))\,k_1} \\
&= \frac{(1-\delta)\,[1 + c_i(1)] + f_k(1) - c_k(1)}{1 + c_i(0)}
\end{aligned}$$

- A functional form.

$$y_t = \theta_t \alpha_k k_t + \alpha_l l_t - \frac{\eta}{2}\left(\frac{i_t}{k_t}\right) i_t$$

$$c_i = \eta\left(\frac{i_t}{k_t}\right); c_k = -\frac{\eta}{2}\left(\frac{i_t}{k_t}\right)^2$$

1. The investment return

$$
\begin{aligned}
R^I_{t+1} &= \frac{(1-\delta)\left[1+c_i(t+1)\right]+f_k(t+1)-c_k(t+1)}{1+c_i(t)} \\
&= \frac{(1-\delta)\left[1+\eta\left(\frac{i_{t+1}}{k_{t+1}}\right)\right]+\theta_t\alpha_k+\frac{\eta}{2}\left(\frac{i_t}{k_t}\right)^2}{1+\eta\left(\frac{i_t}{k_t}\right)} \\
&\approx 1+\theta\alpha_k-\delta+\eta\Delta i_{t+1}
\end{aligned}
$$

2. Intuition. $1+\eta i/k$ are the price terms, and $\theta\alpha$ is the dividend term. If you invest more now, that means prices are high and you don't get much capital for your investment, lowering the return. If you invest more later, that means prices are high, *reducing* capital in the future by \$1 frees up many dollars of investment so the return is good.

   Price terms (P/D,B/M) dominate the variation of stock returns, far more than dividends.

   As Q says "investment should be proportional to stock price" this says "investment growth should be proportional to stock return"

3. Q theory

$$
\begin{aligned}
1+\eta\left(\frac{i_0}{k_0}\right) &= \frac{V}{k_1} = E_0\sum_{t=1}\frac{\Lambda_t}{\Lambda_0}(1-\delta)^{t-1}\left[f_k(t)-c_k(t)\right] \\
&= E_0\sum_{t=1}\frac{\Lambda_t}{\Lambda_0}(1-\delta)^{t-1}\left[\theta_t\alpha_k+\frac{\eta}{2}\left(\frac{i_t}{k_t}\right)^2\right]
\end{aligned}
$$

   Variation in Λ – discount rates – can be the dominant source of variation

4. If we use the standard functional form,

$$
y_t = \theta_t k_t^{\alpha} l_t^{1-\alpha} - \frac{\eta}{2}\left(\frac{i_t}{k_t}\right) i_t
$$

$$
R^I = \frac{(1-\delta)\left[1+\eta\left(\frac{i_{t+1}}{k_{t+1}}\right)\right]+\theta_t\alpha\left(\frac{k_t}{l_t}\right)^{\alpha-1}+\frac{\eta}{2}\left(\frac{i_t}{k_t}\right)^2}{1+\eta\left(\frac{i_t}{k_t}\right)}
$$

   It's easy enough to do. I was just too lazy to get k and l data, since the investment terms completely dominate the series.

Papers

- Paper 1: "Production Based"   This paper exploits the $R^I = R^{\text{stock}}$ prediction for the aggregate market.

1. Note: *Q theory does not allow for an error term.* Q theory is an exact relationship that ought to hold in every time period. There is no expectational error here. Should you run investment on Q or Q on investment? The source of the error is a specification error, and that's hard to make any great assumptions about. This is also true of the stock return = investment return prediction. The "production based" paper looks at a lot of ways that stock returns are correlated with investment returns, but does not offer a reason why the correlation is not perfect.

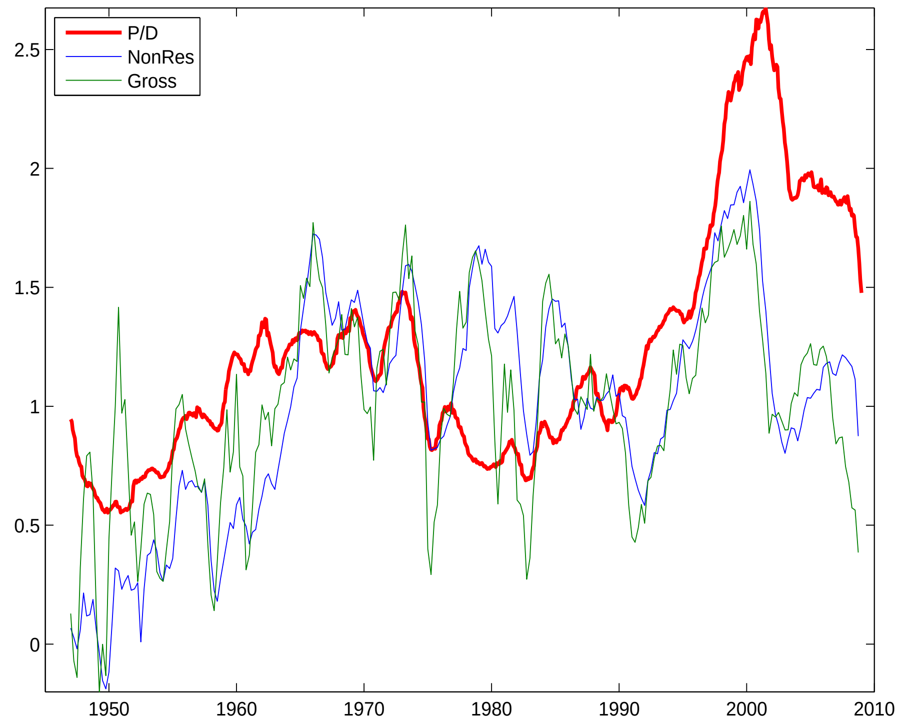
2. The *Return* implication might be easier to use empirically than the *investment-q* relation,

   (a) We look at returns throughout finance rather than price vs. present value. Long term present values are hard to calculate

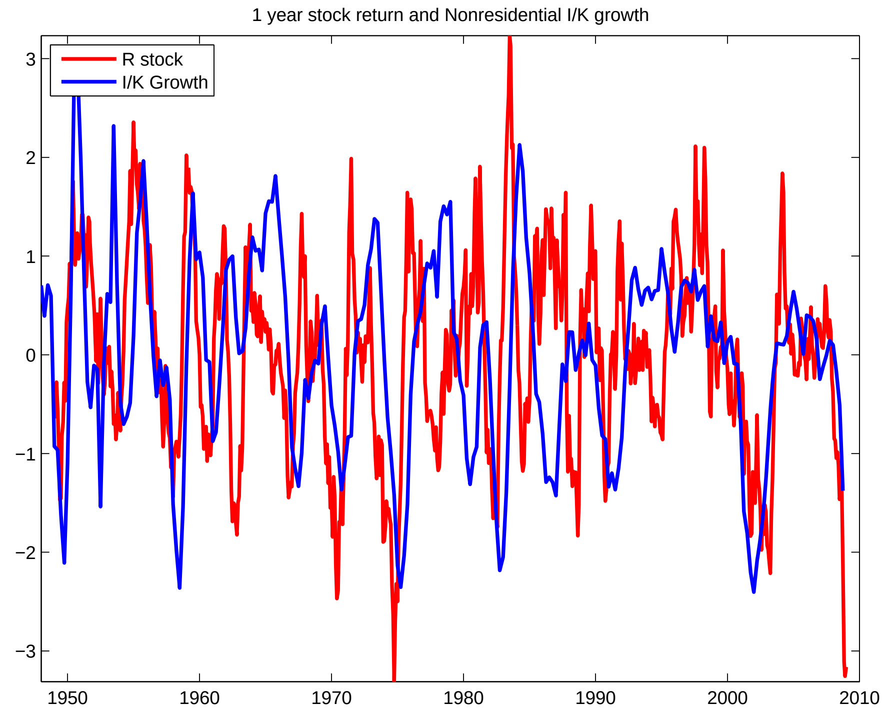
   (b) The *Return* prediction highlights the adjustment cost and the role of investment. The *price* or q prediction highlights proper discounting, modeling of f_k, getting taxes and depreciation allowances right, and so forth.

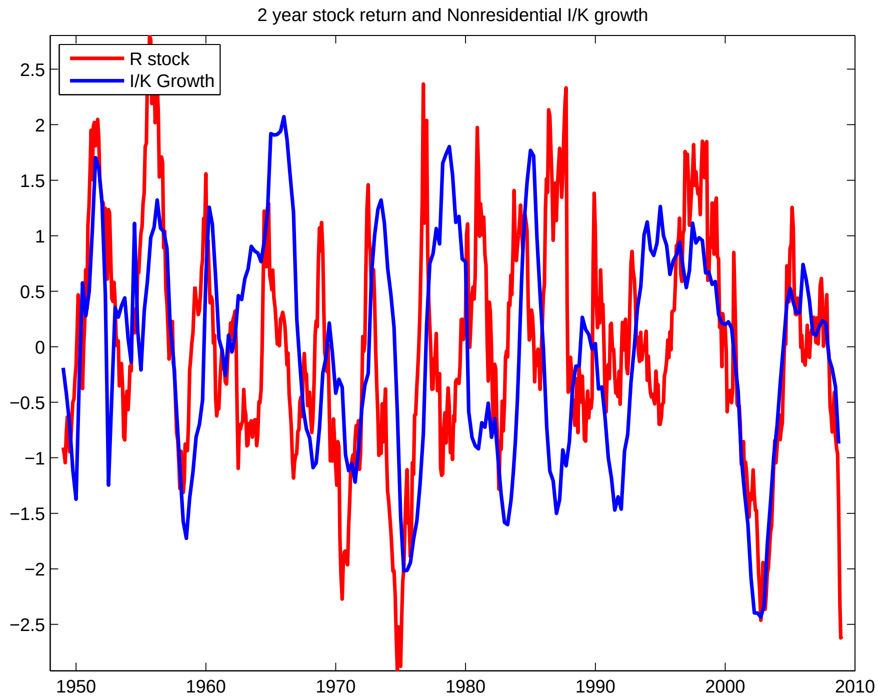
   (c) The return prediction minimizes difficulty in measuring book value. We only need k for i/k, which I imputed from past i. Conventional q measurement spends a lot of time on book value.

3. Some evidence and updated graphs to inspire you. First, let's look at the market P/D (vw crsp) ratio (I can't find Market/Book) and some I/K ratios. To form I/K, I took the real nonresidential, and gross real investment series, and then just accumulated capital based on $\delta = 0.1$. As you can see there is a suggestive correlation! In particular, the Q theory seems to do very well in the 1990s tech boom, bust and the current troubles. The blips line up well, and there is something to the low frequency boom of the 1960s and 1990s and the low frequency movement in P/D. Perhaps a better Market/Book would have less of a trend in it. But you can also see that if this is going to be a model with an R^2 of one, it will be easy to reject, or to have other variables like "cash flow" enter a regression of i on q.



Now, let's look instead at the same data, but this time in "return" form. I graphed the one and two year returns along with the growth $(I_t/K_t)/(I_{t-h}/K_{t-h})$. I standardized both by subtracting the mean and dividing by standard deviation so they would plot together. (I didn't compute an investment return, because I want to convince you there are robust features of the data at work here not needing a complex calculation.)





Those are much better correlations! You can also see that stock returns lead a bit. That might be timing – investment is a quarterly average – and it also might reflect planning lags, which we should incorporate into the technology.

4. Figure 1, timing. You have to deal with stock returns being point to point and investment coming from averages. Given the correlation graphs from Cc simulations, we should be doing a lot more thinking about time aggregation and timing mistakes in any test that involves correlations!
5. Figure 2, correlation between stock and investment returns.
6. Table 3. If $R^I = R^s, E_t(R^I) = E_t(R^S)$.
7. Also notice that I/K forecasts returns! "Companies invest more when the cost of capital is low" makes abundant sense. (This is often cited as an anomaly, for reasons that escape me)
8. Figure 3. $E_t(R^I) = b_I' x_t$ and $E_t(R^S) = b_S' x_t$ (Be careful here, if there were only one x they would look very correlated!)
9. Figure 4. Projection on i/k looks the same. Investment returns are a function of I/K, but stock returns have the same projection.
10. Table V Δy_{t+k} on R should be the same
11. One summary point: q theory may not work well in *levels* but it seems to work pretty well in *first differences*. A low frequency misspecificaiton can ruin levels but leave first differences to work fairly well.

- Lamont, Table 5. *Planned* investment for time t, reported at time $t-1$ does a great job of forecasting returns for time t. This is a good way to start thinking about lags in the investment process.

- Zhang

  1. Claims from the introduction: Q theory explains
     (a) The frequency of equity issuance is procyclical;
     (b) investment is negatively related with future stock returns in the cross section, and the magnitude of this correlation is stronger in firms with higher cash flows;
     (c) firms conducting seasoned equity offerings underperform nonissuers with similar size and book-to-market in the long run;
     (d) the operating performance of issuing firms substantially improves prior to equity offerings, but then deteriorates;
     (e) firms distributing cash back to shareholders outperform other firms with similar size and book-to-market, and the outperformance is stronger in value firms than in growth firms
     (f) relative to industry peers, firms announcing share re-purchases exhibit superior operating performance, but the performance declines following the announcements.
  2. Note marginal $\neq$ average q, so that you still see these effects with b/m controls.

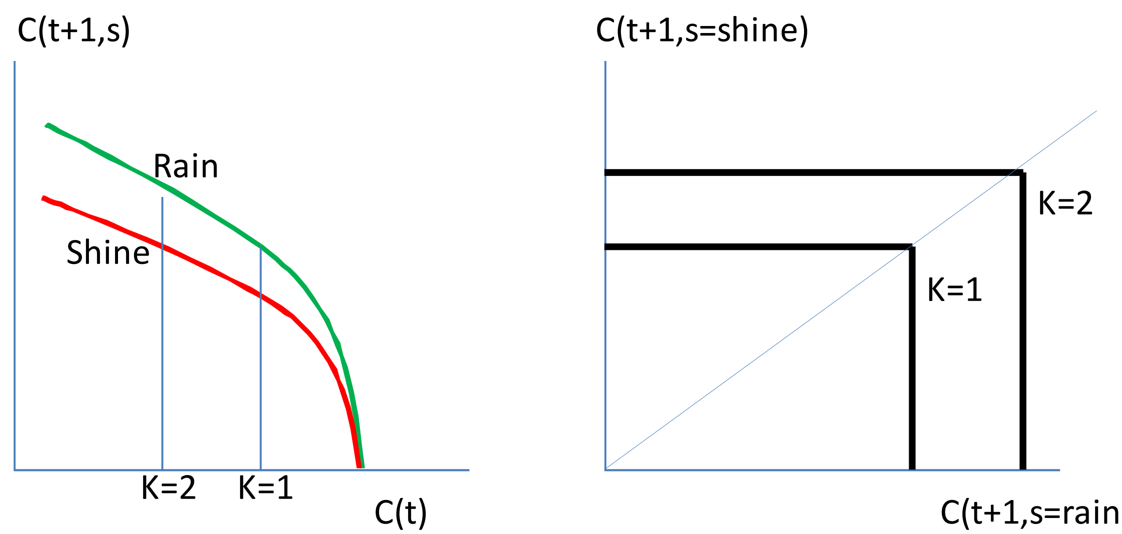
- Comments

  1. Where are the shocks? In a simple q theory analysis here it looks like it's all shocks to preferences – time varying discount rates, leads firms to invest more or less. Why? We know there are strong time varying expected returns, and little time varying expected cash flow ($\theta_t \alpha_k$). In a more sophisticated GE model perhaps news about θ leads to changing risk aversion. Still a GE version of this that answers "where are the shocks" would be welcome.
  2. Limits of "investment return". Often there are lags or irreversibilities in the production process, so that the partial derivative "a little more in today, a little more out tomorrow, *and no other changes in any date and state"* is not physically possible. Of course we can still value margins, "a little more in vs. the present value of the little bit extra we get out at each date/state in the future." And we can still first-difference if there is more signal vs. noise at higher frequencies.

Better theory

- Why no m?

  1. Why did "consumption based" models lead to $m_t = \beta(c_{t+1}/c_t)^{-\gamma}$ but here we only have $1 = E(mR^I)$ or $R^I = R^{\text{stock}}$? Aren't production and consumption supposed to be symmetric? Answer: Look at a very simple technology, $y_{t+1} = \theta_{t+1} f(k_t)$. This allows smooth variation across *time*, but not across *states of nature*.



Here are the production sets implied by $c_{t+1}(s) = \theta_{t+1}(s) f(k_t)$; $k_t + c_t = e_t$. As you can see, the firm can smoothly transform consumption from c_t to c_{t+1} in each state, but this is "joint production" – by investing one dollar, you get more production in *both* states. There is no way to transform consumption from state rain to state shine, or to transform consumption from date t to one state only.

2. Why not? This is a historical accident. Economists started with production sets $y_t = f(k_t)$ that made sense for nonstochastic production theory and nonstochastic growth theory. Then, when we wanted to add randomness, we just tacked a productivity shock $y_t = \theta_t f(k_t)$ on to it. We did *not* get here from deep microeconomic evidence that production sets allow no substitution across states of nature!

3. How can we construct a model $m_{t+1} = f(\text{production data}_{t,t+1})$? Can producers transform across states, and if so how do we model that?

   (a) Cochrane, JPE. just hyptothesizes $m_{t+1} = 1 - b' R^I_{t+1}$. After all, what matters other than the rates of return of investments? I used an APT argument, that other assets had to just be repackaging of available investment returns, or that marginal utility growth had to reflect consumption of those returns. It's pretty ad hoc!

   (b) If we have multiple technologies, we can span multiple states (Jermann's paper expands on this idea).

      1. If the farmer has one field that does well in rainy weather, and one field that does well in sunny weather, then he can transform output from (rain) to (sun) by investing a bit more in the sunny field and a bit less in the rainy field, not changing his overall investment.

      2. This is an ancient idea in finance. *Dynamic trading lets you span a large state space with a few assets.* For example, following Black and Scholes, dynamic trading in "stock" and "bond" lets you span all contingent claims based on the stock outcome, and to construct a discount factor that prices all those states.

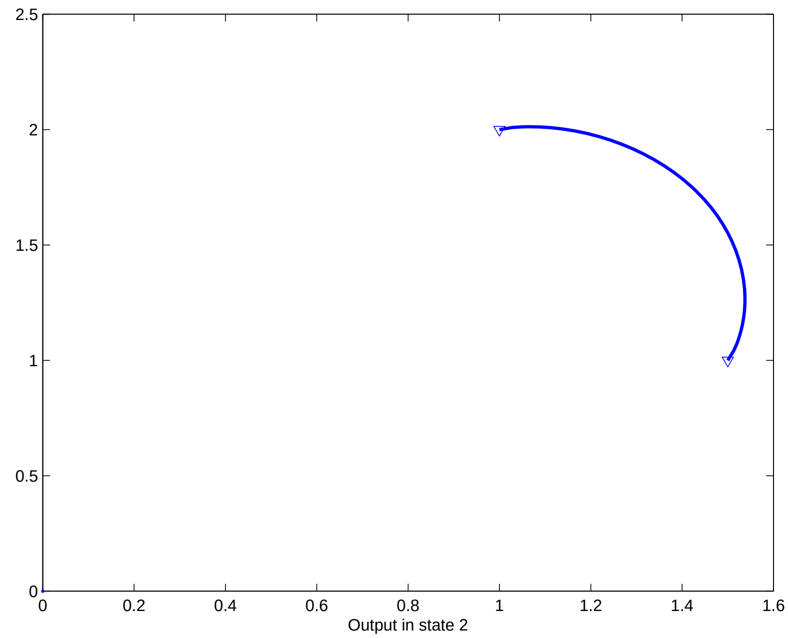
3. A simple example.

$$\begin{aligned} y_1(s) &= \theta_1(s)k_1^\alpha;\ \theta_1 = \left[\begin{array}{cc} 1.5 & 1 \end{array}\right] \\ y_2(s) &= \theta_2(s)k_2^\alpha;\ \theta_2 = \left[\begin{array}{cc} 1 & 2 \end{array}\right] \end{aligned}$$

$$k_1 + k_2 = k$$

$$\begin{aligned} y(s) &= y_1(s) + y_2(s) \\ &= \theta_1(s)k_1^\alpha + \theta_2(s)k_2^\alpha \\ &= \theta_1(s)k_1^\alpha + \theta_2(s)(k-k_1)^\alpha \end{aligned}$$

Using $\alpha = 0.8$, here's what it looks like



The endpoints are where you put all investment into one or the other field. The line between them is convex, because of decreasing returns to scale. Half in each field is, other things equal, better.

4. Once we see where the firm decides to produce, we can now infer contingent claims prices and hence the discount factor!

$$\begin{aligned} y_1(s) &= \theta_1(s)k_1^\alpha; \\ y_2(s) &= \theta_2(s)k_2^\alpha; \end{aligned}$$

$$\max \sum \pi(s)\frac{\Lambda_1(s)}{\Lambda_0}\theta_i(s)k_i^\alpha \text{ s.t. } k_1 + k_2 = k$$

$$\sum \pi(s)\frac{\Lambda_1(s)}{\Lambda_0}\theta_i(s)\alpha k_i^{\alpha-1} = \lambda$$

Note

$$\theta_i(s)\alpha k_i^{\alpha-1} = R_i^I(s) = \alpha y_i(s)/k_i$$

so it's the invetment return, and we can measure it with data on output and capital

$$\left[\begin{array}{cc} \pi(1)\frac{\Lambda_1(1)}{\Lambda_0} & \pi(2)\frac{\Lambda_1(2)}{\Lambda_0} \end{array}\right]\left[\begin{array}{cc} R_1^I(1) & R_2^I(1) \\ R_1^I(2) & R_2^I(2) \end{array}\right] = \left[\begin{array}{cc} \lambda & \lambda \end{array}\right]$$

since we have two returns and two states, we can solve for the discount factor!

$$\left[\begin{array}{cc} \pi(1)\frac{\Lambda_1(1)}{\Lambda_0} & \pi(2)\frac{\Lambda_1(2)}{\Lambda_0} \end{array}\right] = \left[\begin{array}{cc} R_1^I(1) & R_2^I(1) \\ R_1^I(2) & R_2^I(2) \end{array}\right]^{-1}\left[\begin{array}{cc} \lambda & \lambda \end{array}\right]$$

5. This is realistic! *When we aggregate from worker to plant to firm to industry to aggregate production we lose all the capacity to transform across states of nature by changing the composition of investment.*
6. Agenda: start from very disaggregated production functions (a continuum) and describe aggregate production functions across states. Include *dynamic* spanning as well, i.e. varying investment over time. .
7. What do aggregate production sets from this idea look like?

(c) Can we write aggregate production sets directly that allow marginal rates of transformation?

1. Idea:
$$\max_{k,\varepsilon_{t+1}\in\boldsymbol{\theta}} E(m_{t+1}\varepsilon_{t+1}f(k_t)) - k_t = \sum_{s_{t+1}} \pi_s m_s \varepsilon_s f(k) - k$$
2. What's a good $\boldsymbol{\theta}$? *What is a good convex set of random variables*? I tried
$$\left[E\left(\frac{\varepsilon_{t+1}}{\theta_{t+1}}\right)^{\alpha}\right]^{1/\alpha} = \left[\sum \pi_s\left(\frac{\varepsilon_s}{\theta_s}\right)^{\alpha}\right]^{\frac{1}{\alpha}} = 1$$
(note the inspiration from non-expected utility) The idea is, you can choose your productivity shock from a set, increasing the productivity shock in some states, if you reduce it in others. θ represents the "natural" productivity across states.
3. $\partial/\partial\varepsilon$:
$$\begin{aligned} m_s f(k) &= \lambda\frac{\varepsilon_s^{\alpha-1}}{\theta_s^{\alpha}} \\ m_{t+1} &= \lambda\frac{y_{t+1}^{\alpha-1}}{\theta_{t+1}^{\alpha}f(k_t)^{\alpha}} \end{aligned}$$
The first order conditions say to produce more in high contingent claim price states – choose a productivity shock ε that is weighted towards high contingent claim prices. .
4. Empirical trouble: How do you measure θ independently from ε? Frederico Belo has finally made some progress on this idea. .